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An asymptotic model of two-dimensional convection in the limit of low Prandtl number

By F. H. BUSSE AND R. M. CLEVER
Institute of Geophysics and Planetary Physics and Department of Earth and Space Sciences, University of California, Los Angeles, California 90024

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An approximate solution of two-dimensional convection in the limit of low Prandtl number is presented in which the buoyancy force is balanced by the inertial terms. The results indicate that inertial convection becomes possible when the Rayleigh number exceeds a critical value of about 7×10^3. Beyond this value the velocity and temperature fields become independent of the Prandtl number except in thin boundary layers. The convective heat transport approaches the law $Nu = 0{\cdot}175\, R^{\frac{1}{4}}$ for the Nusselt number Nu. These results are in reasonably close agreement with the numerical results described in the preceding paper by Clever & Busse (1980).

1. Introduction

The analytical model described in this paper has been stimulated by the striking demonstration of inertial convection by the numerical computations of Clever & Busse (1980; hereinafter referred to as CB). Convection in a layer heated from below with rigid boundaries is characterized by the development of two different boundary layers as the Rayleigh number is increased. The familiar boundary layer is the thermal boundary layer caused by the fact that the heat flux is carried by convection in the interior of the layer, but requires conduction to cross the boundaries. The no-slip condition at the rigid boundaries is the origin of the development of velocity boundary layers as the characteristic Reynolds number becomes large in comparison with unity. In most laboratory convection experiments, in particular in those with high-Prandtl-number fluids, the Reynolds number stays relatively small and not much attention has been devoted to the velocity boundary layer. Since the Prandtl number P describes the ratio between the thicknesses of the velocity boundary layer and the thermal boundary layer, the former increases in importance as P decreases. Because the non-dimensional thickness of the thermal boundary layer is in first approximation a function of the Rayleigh number R only, it is clear that the cost of numerical computations of convection increases strongly with decreasing P because of the need to resolve the thin velocity boundary layer. Boundary layer models are thus especially desirable for the understanding of low-Prandtl-number convection.

It has been mentioned in CB that the assumption of two-dimensional steady convection is not a realistic one for convection in a low-Prandtl-number fluid, since the transition to time-dependent three-dimensional convection precedes the transition to inertial convection as the Rayleigh number is increased. But inertial convection represents such a highly efficient mode of convection that it must be expected that it

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occurs in a modified form in the case of three-dimensional time dependent convection. The experimental data for the heat transport in low-Prandtl-number fluids strongly support this idea. Thus the analytical model described in the following section is of interest from the physical as well as the mathematical point of view.

It is worth noting at this point, that the onset of three-dimensional convection is delayed when the effects of rotation of the layer about a horizontal axis or the effects of a horizontal uniform magnetic field are considered. It is well known (see, for example, Chandrasekhar 1961) that the dynamics of convection rolls parallel to the axis of rotation or the magnetic field are not influenced by these additional effects, but three-dimensional disturbances are inhibited. The theory presented in this paper thus has a direct application to the case of convection rolls in the presence of a horizontal magnetic field or a horizontal axis of rotation.

2. Mathematical formulation of the problem

We consider a horizontal layer of fluid satisfying the Boussinesq approximation. The temperatures T_1 and T_2 are prescribed at the upper and the lower boundaries of the layer. Using the height d of the layer, d^2/ν, and $T_2 - T_1$ as scales for length, time and temperature, respectively, the non-dimensional equations for the velocity vector $\mathbf{u}$ and the temperature θ assume the form

$$\frac{\partial}{\partial t}\mathbf{u}+\mathbf{u}\cdot\nabla\mathbf{u} = -\nabla\pi+RP^{-1}\mathbf{k}\theta+\nabla^2\mathbf{u}, \tag{2.1a}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2.1b}$$

$$\left(\frac{\partial}{\partial t}\theta+\mathbf{u}\cdot\nabla\theta\right)P = \nabla^2\theta. \tag{2.1c}$$

The Rayleigh number R and the Prandtl number P are defined by

$$R = \frac{\gamma(T_2-T_1)\,gd^3}{\nu\kappa}, \quad P = \frac{\nu}{\kappa}, \tag{2.2}$$

where ν is the kinematic viscosity, κ is the thermometric conductivity, γ is the coefficient of thermal expansion and g is the acceleration of gravity which is directed in the direction opposite to the unit vector $\mathbf{k}$. All terms that can be written as gradients in equation (2.1c) have been combined into $\nabla\pi$. We note that for mathematical convenience the time-scaling used in this paper differs by the factor P from that used in CB.

Using a Cartesian system of co-ordinates with the z co-ordinate in the direction of $\mathbf{k}$ the boundary condition at the rigid top and bottom boundaries can be written in the form

$$\mathbf{u} = 0, \quad \theta = \mp\tfrac{1}{2} \quad \text{at} \quad z = \pm\tfrac{1}{2}. \tag{2.3}$$

The goal of the analysis is to obtain a simple approximation for the solution of equations (2.1) in the case of steady two-dimensional flow. The limit of vanishing Prandtl number will be assumed, but the Rayleigh number will not be subjected to restrictions.

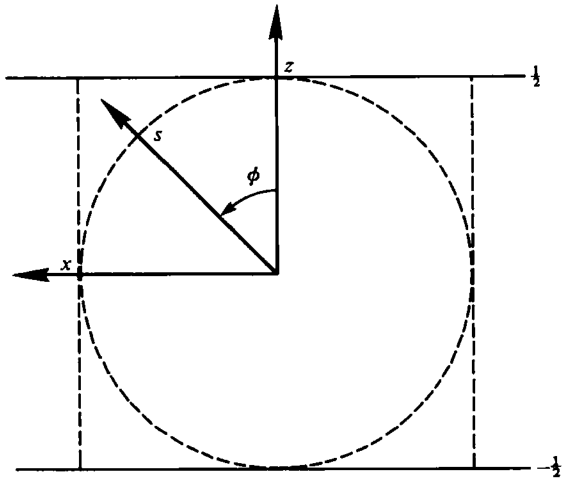

FIGURE 1. Co-ordinate systems for the description of the convection roll.

The assumption of two-dimensional flow allows us to introduce the stream function ψ

$$\mathbf{u} = \nabla \times \mathbf{j}\psi(x, z), \tag{2.4}$$

where $\mathbf{j}$ is the unit vector in the y direction. After taking the y component of the curl of equation (2.1 a) the following equations for ψ and θ are obtained.

$$\nabla^4\psi + RP^{-1}\mathbf{j} \times \mathbf{k}\cdot\nabla\theta = \mathbf{j}\cdot\nabla\psi \times \nabla\nabla^2\psi, \tag{2.5a}$$

$$\nabla^2\theta = P\mathbf{j}\cdot\nabla\theta \times \nabla\psi. \tag{2.5b}$$

In the following, equation (2.5b) will be solved first after assuming a special stream function ψ. Then equation (2.5a) will be solved and the arbitrary parameter introduced in the analysis will be determined.

3. An Oseen approximation of the heat equation

The analysis of this section is based on the assumption that two-dimensional convection rolls in a low-Prandtl-number fluid exhibit the 'flywheel'-character first demonstrated by Jones, Moore & Weiss (1976) and found by CB in the case of an infinite layer heated from below. Accordingly, ψ depends only on the distance s from the centre-line of each roll. Introducing the polar system of co-ordinates

$$s = z\cos\phi + x\sin\phi, \quad \phi = \arctan x/z, \tag{3.1}$$

we restrict the attention to a single convection roll as shown in figure 1. It is evident from this figure that the assumption of 'flywheel' convection implies that the wavelength of convection exceeds twice the depth of the layer. For simplicity we shall restrict the attention to the case when the wavelength is about twice the thickness of the layer as in the case of the critical value of the wavenumber. In order to simplify the problem further, we shall assume that ψ can be approximated by

$$\psi_0 = V(1-4s^2)/8P \quad \text{for} \quad s \leqslant \tfrac{1}{2}, \tag{3.2}$$

i.e. we shall solve the equation (2.5*b*) under the assumption that the motion corresponds to a rigid rotation. This represents an Oseen approximation which should yield reasonable results as long as V corresponds to a suitably defined average of the function $\psi(s)$.

After introducing definitions (3.1) and (3.2) equation (2.5*b*) assumes the form

$$\nabla^2\theta = V\frac{\partial}{\partial\phi}\theta. \tag{3.3}$$

The general solution which is regular at the origin is readily obtained in terms of the modified Bessel functions of the first kind,

$$\theta = \mathscr{R}\left(\sum_{m=1}^{\infty}\frac{I_m(s\sqrt{(iV)})}{I_m(\frac{1}{2}\sqrt{(iV)})}\theta_m e^{im\phi}\right) \quad \text{for} \quad s \leqslant \tfrac{1}{2}, \tag{3.4}$$

where the assumption has been made that the y dependence of θ at $s = \frac{1}{2}$ is given by

$$\theta = \mathscr{R}\left(\sum_{m=1}^{\infty}\theta_m e^{im\phi}\right) \quad \text{at} \quad s = \tfrac{1}{2} \tag{3.5}$$

$\mathscr{R}$ denotes the real part in these expressions while $\mathscr{I}$ will be used for the imaginary part. The coefficients θ_m depend on the details of the solution of equation (2.5*b*) in the regions with $s > \frac{1}{2}$ and on the boundary condition (2.3). For the analysis of the following sections only the coefficient θ_1 is important. Assuming that the solution $\theta = -s\cos\phi$ of the static problem provides a reasonable basis for the determination of θ_1 we obtain

$$\theta_1 = -\tfrac{1}{2}. \tag{3.6}$$

The results derived in the following sections do not change much if a small imaginary part is added on the right-hand side of (3.6).

4. Solution of the equation of motion

When expression (3.4) is introduced into equation (2.5*a*) and the assumption is made that the component of ψ proportional to $\exp\{i2\phi\}$ is negligible in comparison to the mean component $\psi(s)$, the equation

$$\begin{aligned}\left(\frac{1}{s}\frac{d}{ds}s\frac{d}{ds}\right)^2\psi(s) &= -\overline{\left(\sin\phi\frac{\partial}{\partial s}\theta+\cos\phi\frac{\partial}{s\partial\phi}\theta\right)}RP^{-1}\\ &= -\tfrac{1}{4}RP^{-1}\mathscr{I}\left\{\sqrt{(iV)}\left(I_1'(s\sqrt{(iV)})+\frac{1}{s\sqrt{(iV)}}I_1(s\sqrt{(iV)})\right)\Big/I_1(\tfrac{1}{2}\sqrt{(iV)})\right\}\\ &= -\tfrac{1}{4}RP^{-1}\mathscr{I}\left\{\frac{\sqrt{(iV)}I_0(s\sqrt{(iV)})}{I_1(\frac{1}{2}\sqrt{(iV)})}\right\}\end{aligned} \tag{4.1}$$

is obtained where the bar indicates the average over the ψ dependence. Integration of (4.1) yields

$$\frac{1}{s}\frac{d}{ds}\left(s\frac{d}{ds}\psi(s)\right) = -\tfrac{1}{4}RP^{-1}\mathscr{I}\left\{\frac{I_0(s\sqrt{(iV)})}{\sqrt{(iV)}I_1(\frac{1}{2}\sqrt{(iV)})}\right\}+W_0 \tag{4.2}$$

and by further integration we find

$$\psi(s) = -\tfrac{1}{4}RP^{-1}\mathscr{I}\left\{\frac{I_0(s\sqrt{(iV)})}{(iV)^{\frac{3}{2}}I_1(\frac{1}{2}\sqrt{(iV)})}\right\} - W_0(1-4s^2)/8, \tag{4.3}$$

where the condition of regularity at the origin has been imposed. In the appendix it is shown that the appropriate boundary condition in the limit $P \to 0$ is given by

$$\frac{d}{ds}\psi(s) = 0 \quad \text{at} \quad s = \tfrac{1}{2}. \tag{4.4}$$

Application of this condition yields

$$W_0 = \tfrac{1}{2}RP^{-1}\mathscr{I}\left\{\frac{I_0'(\frac{1}{2}\sqrt{(iV)})}{(iV)\,I_1(\frac{1}{2}\sqrt{(iV)})}\right\}. \tag{4.5}$$

The convection velocity u_ϕ is thus given by

$$u_\phi = \frac{d}{ds}\psi = \tfrac{1}{4}RP^{-1}\mathscr{R}\left\{\frac{I_0'(s\sqrt{(iV)}) - 2sI_0'(\frac{1}{2}\sqrt{(iV)})}{VI_1(\frac{1}{2}\sqrt{(iR)})}\right\}$$

$$= \tfrac{1}{4}R(PV)^{-1}\mathscr{R}\left\{\frac{I_1(s\sqrt{(iV)})}{I_1(\frac{1}{2}\sqrt{(iV)})} - 2s\right\}. \tag{4.6}$$

The right-hand side can be easily evaluated in the limits of low and high Peclet number V by using the relationships (see, for example, Abramowitz & Stegun 1965),

$$\text{for } V \ll 1, \quad I_1(s\sqrt{(iV)}) = \tfrac{1}{2}s\sqrt{(iV)}\left[1 + \frac{iVs}{4\cdot 2!} + \frac{(iVs^2)^2}{16\cdot 3!\cdot 2!} + \ldots\right]; \tag{4.7a}$$

$$\text{for } V \gg 1, \quad I_1(s\sqrt{(iV)}) = \exp\{s\sqrt{(iV)}\}\,(1 - 3/8s\sqrt{(iV)} + \ldots)/(4\pi^2 s^2 iV)^{\frac{1}{4}}. \tag{4.7b}$$

An approximate solution for the combined equations (2.5) is obtained when the expression for u_ϕ following from (3.2) is related to expression (4.6).

In the spirit of the Oseen approximation we equate the mean angular velocity of the velocity field (4.6) with the constant VP^{-1} of expression (3.2). In the low-Peclet-number limit (4.7*a*) we obtain

$$VP^{-1} \approx RVP^{-1}(3072 + V^2)^{-1}\, 8\int_0^{\frac{1}{2}}(1 - 6s^2 + 8s^4)\,s\,ds, \tag{4.8}$$

where terms of the order V^3 and of higher order have been neglected. Relationship (4.8) yields

$$V^2 \approx \tfrac{5}{12}R - 3072, \tag{4.9}$$

which indicates that inertial convection can be expected only when the Rayleigh number exceeds a second critical value of the order 7×10^3,

$$R_2 \approx 12\cdot 3072/5 \approx 7373. \tag{4.10}$$

For large values of the Peclet number V for which the Oseen approximation is especially appropriate we obtain in place of (4.8)

$$VP^{-1} \approx \tfrac{1}{2}R(PV)^{-1}, \tag{4.11}$$

which indicates that the Peclet number V grows asymptotically proportionally to the square root of the Rayleigh number.

5. Discussion

The numerical computations of the Nusselt number in low-Prandtl-number fluids shown in figure 2 of CB strongly suggest the existence of a second critical Rayleigh number R_2 at which the slope of the heat transport changes discontinuously in the limit of vanishing Prandtl number. But it is difficult to extrapolate the available numerical results to that limit. The data are certainly not in disagreement with a value of R_2 of the order 7×10^3.

A mathematical test of the Oseen approximation can be deduced from the analysis of Proctor (1977) who solved equations (2.5) in the low-Peclet-number limit. Although Proctor's work dealt with the onset of inertial convection in a horizontal cylindrical tube heated from below, the boundary conditions used in his analysis turn out to be identical to those applied in the present model. Because of the different scaling used in Proctor's paper, his value of the second critical Rayleigh number must be multiplied by 16 for the comparison with the present analysis. After slightly correcting the numerical values given in Proctor's paper we obtain

$$R_2 = 6900,$$

which is in reasonable agreement with the Oseen value (4.10).

Of special interest is the asymptotic expression for the heat transport. According to the basic equations (2.1) the ratio between the heat transport by convection alone and the heat flux in the absence of convection is given by

$$Nu - 1 = P\langle \theta \mathbf{k} . \nabla \times \mathbf{j}\psi \rangle, \tag{5.1}$$

where the angular brackets indicate the average over the fluid layer and Nu denotes the Nusselt number. Using the asymptotic expression for large Peclet numbers V (4.7b) in evaluating (3.6a) and (4.6), the right-hand side of (5.1) yields

$$Nu - 1 = \frac{3\pi}{64\sqrt{(2)}} RV^{-\frac{3}{4}} \approx \frac{3\pi}{64} (2R)^{\frac{1}{4}}. \tag{5.2}$$

This asymptotic expression for the Nusselt number lies slightly above the numerical values computed for low Prandtl numbers in figure 2 of CB, but agrees well with the asymptotic slope indicated by the curves. The remarkable property of the relationship (5.2) is its independence of the Prandtl number which contrasts sharply with the small-amplitude perturbation result of Schlüter, Lortz & Busse (1965) and also with the viscosity independent law $Nu \propto (RP)^{\frac{1}{2}}$ often used in astrophysical applications.

Relationship (5.2) also compares well with the experimental measurements of Rossby (1969) in mercury ($P = 0{\cdot}025$) which yield

$$Nu \approx 0{\cdot}147\, R^{0{\cdot}257 \pm 0{\cdot}004}.$$

Even though the convection flow is three dimensional and time dependent under the conditions of the experiment, the remarkably high observed heat transport indicates that the realized convection flow possesses the same basic properties as the inertial convection mode discussed in this paper.

An asymptotic relationship similar to (5.2) can be derived for the kinetic energy of convection in the double limit of large R and small P. Using the large-Peclet-number limit of (4.6) we obtain

$$\tfrac{1}{2}\langle \mathbf{u}.\mathbf{u}\rangle = \pi(R/16PV)^2 \approx R\pi/128P^2, \tag{5.3}$$

which agrees well with the high-Rayleigh-number values shown in figure 11 of CB. Note, that the non-dimensional velocity of CB differs by a factor P from that used in this paper. Again, the remarkable fact is that the convective velocity scales with the velocity of thermal diffusion even in the limit when the latter tends to infinity at a given value of the viscosity.

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Appendix

Since the boundary conditions acting on the convection roll velocity field depend strongly on the ϕ co-ordinate introduced in (3.1), it seems at first sight unlikely that a circular motion depending only on the s co-ordinate could give a valid approximation. But the viscous boundary layer influenced by the non-axisymmetric component of tangential stress has a characteristic thickness given by

$$\delta = d(VP/R)^{\frac{1}{2}},$$

which tends to zero in the limit of vanishing Prandtl number. In other words, because of its high Reynolds number the convection flow is affected essentially only by the average tangential stress acting on its boundary.

In order to investigate the problem of the proper boundary condition in more detail, we follow the analysis of Burggraf (1966). We consider a circular eddy, the velocity of which is approximately described by the stream function $\psi = \psi(s)$. While it can be assumed that ψ satisfies the condition

$$\psi = 0 \quad \text{at} \quad s = \tfrac{1}{2}, \tag{A 1}$$

a perturbation stream function $\tilde{\psi}$ must be added in order to satisfy the ϕ-dependent viscous boundary conditions imposed at $s = \frac{1}{2}$

$$\hat{\psi} = \psi + \tilde{\psi}.$$

By taking the y component of the curl of the equations of motion for $\hat{\psi}$ we obtain

$$\nabla^4\tilde{\psi} = -\frac{\partial}{\partial\phi}\nabla^2\tilde{\psi}\left(\frac{1}{s}\frac{\partial}{\partial s}\psi\right), \tag{A 2}$$

where terms quadratic in $\tilde{\psi}$ have been neglected. Introducing the Oseen approximation we replace $s^{-1}\partial\psi/\partial s$ by its average value $-W$,

$$\nabla^4\tilde{\psi} = W\frac{\partial}{\partial\phi}\nabla^2\tilde{\psi}. \tag{A 3}$$

The general solution of this equation which is regular at the origin is given by

$$\tilde{\psi} = a_0 - b_0 s^2 + \sum_{n=1}^{\infty} [a_n s^n + b_n I_n(s\sqrt{(inW)})] \exp\{in\phi\}. \tag{A 4}$$

The condition $\tilde{\psi} = 0$ at $s = \frac{1}{2}$ requires

$$b_0 = 4a_0, \quad b_n = -a_n I_n(\tfrac{1}{2}\sqrt{(inW)}).$$

We now consider the special case of the boundary conditions

$$\frac{\partial}{\partial s}\hat{\psi} = \frac{\partial}{\partial s}\psi + \frac{\partial}{\partial s}\tilde{\psi} = 0 \quad \text{for} \quad \phi = 0, \quad \pi \quad \text{at} \quad s = \tfrac{1}{2}, \tag{A 5}$$

$$\frac{\partial}{\partial s}\left(\frac{1}{s}\frac{\partial}{\partial s}\hat{\psi}\right) = \frac{\partial}{\partial s}\frac{1}{s}\left(\frac{\partial}{\partial s}\psi + \frac{\partial}{\partial s}\tilde{\psi}\right) = 0 \quad \text{for} \quad \phi = \tfrac{1}{2}\pi, \quad \tfrac{3}{2}\pi \quad \text{at} \quad s = \tfrac{1}{2}, \tag{A 6}$$

with some smooth interpolation between the two conditions for intermediate angles ϕ. In first approximation all terms in expression (A 4) can be neglected except for those corresponding to $n = 2$. In the limit of high Reynold number,

$$W \to \infty, \tag{A 7}$$

the asymptotic representation for the modified Bessel functions $I_n(x)$ may be used. Accordingly, we set

$$\tilde{\psi} \approx a_2[s^2 - \exp\{(s-\tfrac{1}{2})\sqrt{(iW)}\}/4(2s)^{\frac{1}{2}}]\exp\{2i\phi\} \tag{A 8}$$

and obtain from (A 6)

$$a_2 = -\frac{4}{W}\left(\frac{\partial}{\partial s}\frac{1}{s}\frac{\partial}{\partial s}\psi\right) + \dots, \tag{A 9}$$

where terms of the order W^{-2} have been neglected. From expression (A 9) follows that in the limit (A 7), the $\tilde{\psi}$ term in (A 5) is negligible, and boundary condition (A 5) reduces to

$$\frac{\partial}{\partial s}\psi = 0 \quad \text{at} \quad s = \tfrac{1}{2}. \tag{A 10}$$

The boundary conditions for the problem described in figure 1 are more complicated than those considered here and in general all coefficients corresponding to an even subscript n in expression (A 4) must be taken into account. But it is unlikely that this will change the boundary condition (A 10) in the limit (A 7) which is the appropriate one for the problem considered in § 4.

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